5 The Dirac Equation and Spinors

In this section we develop the appropriate wavefunctions for fundamental fermions and bosons.

5.1 Notation Review

The three dimension differential operator is $\vec{\nabla}$:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{5.1}$$

We can generalise this to four dimensions ∂_μ:

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \tag{5.2}$$

5.2 The Schrödinger Equation

First consider a classical *non-relativistic* particle of mass m in a potential U. The energy-momentum relationship is:

$$E = \frac{p^2}{2m} + U \tag{5.3}$$

we can substitute the differential operators:

$$\hat{E} \to i\hbar\frac{\partial}{\partial t} \qquad \hat{p} \to -i\hbar\vec{\nabla} \tag{5.4}$$

to obtain the non-relativistic **Schrödinger Equation** (with $\hbar = 1$):

$$i\frac{\partial \psi}{\partial t} = \left(-\frac{1}{2m}\vec{\nabla}^2 + U\right)\psi \tag{5.5}$$

For $U = 0$, the free particle solutions are:

$$\psi(\vec{x}, t) \propto e^{-iEt}\psi(\vec{x}) \tag{5.6}$$

and the probability density ρ and current $\vec{j}$ are given by:

$$\rho = |\psi(x)|^2 \qquad \vec{j} = -\frac{i}{2m}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) \tag{5.7}$$

with conservation of probability giving the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0, \tag{5.8}$$

Or in Covariant notation:

$$\partial_\mu j^\mu = 0 \ \text{ with } \ j^\mu = (\rho, \vec{j}) \tag{5.9}$$

The Schrödinger equation is 1st order in $\partial/\partial t$ but second order in $\partial/\partial x$. However, as we are going to be dealing with relativistic particles, space and time should be treated equally.

5.3 The Klein-Gordon Equation

For a *relativistic particle* the energy-momentum relationship is:

$$p \cdot p = p_\mu p^\mu = E^2 - |\vec{p}|^2 = m^2 \tag{5.10}$$

Substituting the equation (5.4), leads to the relativistic **Klein-Gordon equation**:

$$\left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2\right)\psi = m^2\psi \tag{5.11}$$

The free particle solutions are plane waves:

$$\psi \propto e^{-ip\cdot x} = e^{-i(Et-\vec{p}\cdot\vec{x})} \tag{5.12}$$

The Klein-Gordon equation successfully describes **spin 0 particles** in relativistic quantum field theory.

There are problems with the interpretation of the positive and negative energy solutions of the Klein-Gordon equation, $E = \pm\sqrt{p^2 + m^2}$, since the negative energy solutions have negative probability densities ρ.

5.4 The Dirac Equation

The problems with the Klein-Gordon equation led Dirac to search for an alternative relativistic wave equation in 1928, in which the time and space derivatives are first order. The **Dirac equation** can be thought of in terms of a "square root" of the Klein-Gordon equation. In covariant form it is written:

$$\left(i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi = 0 \qquad (i\gamma^\mu\partial_\mu - m)\,\psi = 0 \tag{5.13}$$

where we have introduced the coefficients $\gamma^\mu = (\gamma^0, \vec{\gamma}) = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$, which have to be determined.

As we will see in equation (5.20), the Dirac equation is simply four coupled differential equations, describing a wavefunction ψ with four components.

5.5 The Gamma Matrices

To find what the γ^μ, $\mu = 0, 1, 2, 3$ objects are, we first multiply the Dirac equation by its conjugate equation:

$$\psi^\dagger\left(-i\gamma^0\frac{\partial}{\partial t} - i\vec{\gamma}\cdot\vec{\nabla} - m\right)\left(i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} - m\right)\psi = 0 \tag{5.14}$$

and demand that this be consistent with the Klein-Gordon equation, (5.11). This leads to the following conditions on the γ^μ:

$$\begin{gathered}(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1 \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \text{ for } \mu \neq \nu \\ \text{with } i = 1, 2, 3,\ \mu, \nu = 0, 1, 2, 3\end{gathered} \tag{5.15}$$

Equivalently in terms of anticommutation relations and the metric tensor (equation (3.3)):

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu, \gamma^\nu + \gamma^\nu, \gamma^\mu = 2g^{\mu\nu} \qquad \mu, \nu = 0, 1, 2, 3 \tag{5.16}$$

The simplest solution for the γ^μ, that satisfies these anticommutation relations, are 4×4 **unitary matrices**. We will use the following representation for the γ matrices:

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix} \tag{5.17}$$

where $\mathbf{I}$ denotes a 2×2 identity matrix, $\mathbf{0}$ denotes a 2×2 null matrix, and the σ^i are the **Pauli spin matrices**:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.18}$$

Let's write out the gamma matrices in full:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{5.19}$$

Please note, despite the μ superscript, the γ^μ are not four vectors. However they do remain constant under Lorentz transforms.

Finally let's write out the Dirac Equation in full:

$$\begin{pmatrix} i\frac{\partial}{\partial t} - m & 0 & i\frac{\partial}{\partial z} & i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \\ 0 & i\frac{\partial}{\partial t} - m & i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & -i\frac{\partial}{\partial z} \\ -i\frac{\partial}{\partial z} & -i\frac{\partial}{\partial x} - \frac{\partial}{\partial y} & -i\frac{\partial}{\partial t} - m & 0 \\ -i\frac{\partial}{\partial x} + \frac{\partial}{\partial y} & i\frac{\partial}{\partial z} & 0 & -i\frac{\partial}{\partial t} - m \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{5.20}$$

5.6 Spinors

The Dirac equation describes the behaviour of spin-1/2 fermions in relativistic quantum field theory. For a free fermion the wavefunction is the product of a plane wave and a **Dirac spinor**, $u(p^\mu)$:

$$\psi(x^\mu) = u(p^\mu)e^{-ip\cdot x} \tag{5.21}$$

Substituting the fermion wavefunction, ψ, into the Dirac equation:

$$(\gamma^\mu p_\mu - m)u(p) = 0 \tag{5.22}$$

For a particle at rest, $\vec{p} = 0$, we find the following equations:

$$\left(i\gamma^0 \frac{\partial}{\partial t} - m\right)\psi = \left(\gamma^0 E - m\right)\psi = 0 \qquad \hat{E}\,u = \begin{pmatrix} m\mathbf{I} & 0 \\ 0 & -m\mathbf{I} \end{pmatrix} u \tag{5.23}$$

The solutions are four eigenspinors:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad u^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.24}$$

and the associated wavefunctions of the fermion is:

$$\psi^1 = e^{-imt}u^1 \qquad \psi^2 = e^{-imt}u^2 \qquad \psi^3 = e^{+imt}u^3 \qquad \psi^4 = e^{+imt}u^4 \tag{5.25}$$

Note that the spinors are 1×4 column matrices, and that there are four possible states. The spinors are, however, not four-vectors: the four components do not represent t, x, y, z.

The four components are a suprise: we would expect only two spin states for a spin-1/2 fermion! Note also the change of sign in the exponents of the plane waves in the states ψ^3 and ψ^4. The four solutions in equations (5.24) and (5.25) describe two different spin states ($\uparrow$ and $\downarrow$) with $E = m$, and two spin states with $E = -m$.

5.7 Negative Energy Solutions & Antimatter

To describe the negative energy states, Dirac postulated that an electron in a positive energy state is produced from the vacuum accompanied by a *hole* with negative energy. The hole corresponds to a physical **antiparticle**, the positron, with charge $+e$.

Another interpretation (Feynman-Stückelberg) is that the $E = -m$ solutions can either describe a negative energy particle which propagates backwards in time, or a positive energy antiparticle propagating forward in time:

$$e^{-i[(-E)(-t)-(-\vec{p})\cdot(-\vec{x})]} = e^{-i[Et-\vec{p}\cdot\vec{x}]} \tag{5.26}$$

5.8 Spinors for Moving Particles

For a moving particle, $\vec{p} \neq 0$ the Dirac equation becomes (using (5.13) and (5.17)):

$$(\gamma^\mu p_\mu - m)\begin{pmatrix} u_A\; u_B \end{pmatrix} = \begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E-m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0 \tag{5.27}$$

where u_A and u_B denote the 1×2 upper and lower components of u respectively. The equations for u_A and u_B are coupled:

$$u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}u_B \qquad u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}u_A \tag{5.28}$$

The solutions are obtained by successively setting: $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, to give:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ (p_x + ip_y)/(E+m) \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \end{pmatrix}$$
$$u^3 = \begin{pmatrix} -p_z/(-E+m) \\ (-p_x - ip_y)/(-E+m) \\ 1 \\ 0 \end{pmatrix} \quad u^4 = \begin{pmatrix} (-p_x + ip_y)/(-E+m) \\ p_z/(-E+m) \\ 0 \\ 1 \end{pmatrix} \tag{5.29}$$

The u^1 and u^2 solutions describe an electron of energy $E = +\sqrt{m^2 + \vec{p}^2}$, and momentum $\vec{p}$:

$$\psi = u^1(p^\mu)e^{-ip\cdot x} \qquad \psi = u^2(p^\mu)e^{-ip\cdot x} \tag{5.30}$$

The u^3 and u^4 of equation (5.29) describe a positron of energy $E = -\sqrt{m^2 + \vec{p}^2}$, and momentum $\vec{p}$.

It is usual to change to the spinors $v^2(p) \equiv u^3(-p)$ and $v^1(p) \equiv u^4(-p)$ to describe these *positive* energy antiparticle states, $E = +\sqrt{m^2 + \vec{p}^2}$

$$v^2(p^\mu) \equiv u^3(-p^\mu) = \begin{pmatrix} p_z/(E+m) \\ (p_x + ip_y)/(E+m) \\ 1 \\ 0 \end{pmatrix} \qquad \psi = v^2(p^\mu)e^{-ip\cdot x} = u^3(-p^\mu)e^{i(-p)\cdot x}$$

$$v^1(p^\mu) \equiv u^4(-p^\mu) = \begin{pmatrix} (p_x - ip_y)/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix} \qquad \psi = v^1(p^\mu)e^{-ip\cdot x} = u^4(-p^\mu)e^{i(-p)\cdot x} \tag{5.31}$$

The u and v are the solutions of:

$$(i\gamma^\mu p_\mu - m)u = 0 \qquad (i\gamma^\mu p_\mu + m)v = 0 \tag{5.32}$$

5.9 Spin and Helicity

The two different solutions for each of the fermions and antifermions corresponds to two possible spin states. For a fermion with momentum $\vec{p}$ along the z-axis, $\psi = u^1(p^\mu)e^{-ip\cdot x}$ describes a spin-up fermion and $\psi = u^2(p^\mu)e^{-ip\cdot x}$ describes a spin-down fermion. For an antifermion with momentum $\vec{p}$ along the z-axis, $\psi = v^1(p^\mu)e^{-ip\cdot x}$ describes a spin-up antifermion and $\psi = v^2(p^\mu)e^{-ip\cdot x}$ describes a spin-down antifermion.

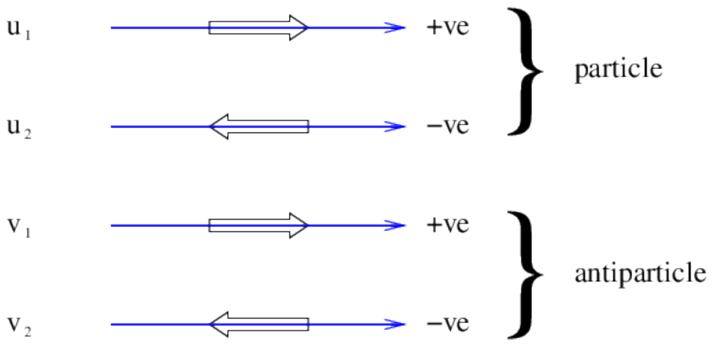


Figure 5.1: Helicity eigenstates for a particle or antiparticle travelling along the $+z$ axis.

The u^1, u^2, v^1, v^2 spinors are only eigenstates of $\hat{S}_z$ for momentum $\vec{p}$ along the z-axis. There's nothing special about projecting out the component of spin along the z-axis, that's just the conventional choice. For our purposes it makes more sense to project the spin along the particle's direction of flight, this defines the **helicity**, h of the particle.

$$\hat{h} = \frac{\vec{S}\cdot\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}\cdot\vec{p}}{|\vec{p}|} \tag{5.33}$$

For a spin-1/2 fermion, the two possible values of h are $h = +1$ or $h = -1$. We call $h = +1$ **right-handed** and $h = -1$ **left-handed**.

The possible states of particles and antiparticles are shown in Figure 5.1. As we will see, the concept of left- and right-handedness plays an important role in calculating matrix elements and in the weak force.

If it also worth noting here, massless fermions, are purely left-handed (only u^2); massless antifermions are purely right handed (only v^1).

5.10 Projection Operators

Helicity is not a Lorentz invariant quantity, therefore we also define a related Lorentz invariant quantity: **chirality**.

Chirality can be defined in terms for the **chiral projection operators**, P_L and P_R:

$$P_L = \frac{1}{2}(1-\gamma_5) \qquad P_R = \frac{1}{2}(1+\gamma_5) \tag{5.34}$$

where γ_5 is another 4×4 matrix:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \qquad (\gamma^5)^2 = 1 \qquad \{\gamma^5, \gamma^\mu\} = 0 \tag{5.35}$$

P_L and P_R project out the left-handed and right-handed chiral components of a spinor:

$$u_L = P_L u \quad u_R = P_R u \tag{5.36}$$

For the two antifermions states remember that the direction of the momentum was reversed in going from the u to the v spinors. Hence the projections of the v spinors are:

$$v_R = P_L v \quad v_L = P_R v \tag{5.37}$$

In the highly relativistic limit, $E \gg m$, $\beta \to 1$ the left-handed chiral states and the same as the left-handed helicity states, and similarly for the right-handed states.

Setting $m = 0$ and $p = p_z = E$ in the highly relativistic limit $\beta \to 1$:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad v^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad v^1 = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \tag{5.38}$$

In a future lecture we'll see that the weak $W^{\pm}$ couplings contain $P_L = (1 - \gamma^5)/2$, and hence only couple to left-handed particles or right-handed antiparticles.

6 Quantum Electrodynamics

6.1 Notation: the Metric Tensor

In this section we can't avoid using the metric tensor, equation (3.3):

$$g_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{6.1}$$

6.2 Fermion currents

We need to define a Lorentz invariant quantity to describe fermion currents for QED. We define the **adjoint spinor** $\bar{\psi} \equiv \psi^\dagger \gamma^0$, where $\psi^\dagger$ is the hermitian conjugate (complex conjugate transpose) of ψ:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \psi^\dagger = (\psi^*)^T = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \qquad \bar{\psi} \equiv \psi^\dagger \gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*) \tag{6.2}$$

The adjoint Dirac equation can be formed by taking the hermitian conjugate of the Dirac equation (5.13), and multiplying it from the right by γ^0:

$$i\partial_\mu \bar{\psi} \gamma^\mu + m\bar{\psi} = 0 \tag{6.3}$$

Multiplying the adjoint Dirac equation (6.3) by ψ from the right, (or the original Dirac equation by $\bar{\psi}$ from the left) gives the continuity equation (c.f. equation **??**equ:continuity)):

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = \bar{\psi}\gamma^\mu(\partial_\mu\psi) + (\partial_\mu\bar{\psi})\gamma^\mu\psi = 0 \quad \text{or} \quad \partial_\mu j^\mu = 0 \tag{6.4}$$

Where j^μ is the four-vector fermion current:

$$j^\mu = \bar{\psi}\gamma^\mu\psi = (\bar{\psi}\gamma^0\psi, \bar{\psi}\vec{\gamma}\psi) = (\rho, \vec{j}) \tag{6.5}$$

and ρ is the probability density:

$$\rho = j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi \tag{6.6}$$

The fermion current $j^\mu = \bar{\psi}\gamma^\mu\psi$ and is has the properties of a Lorentz four-vector, which is what we required. Additionally, the probability density ρ is positive definite for all four possible spinor states. This is only true if we use the *adjoint* form with $\bar{\psi}$.

6.3 Maxwell's Equations

The photon is described by Maxwell's equations.

For electromagnetic interactions, Maxwell's equations can be written in a Lorentz covariant form:

$$\partial^2 A^\mu = 4\pi j^\mu \qquad \partial_\mu A^\mu = 0 \qquad \partial_\mu j^\mu = 0 \tag{6.7}$$

where $\partial^2 = \partial_\nu \partial^\nu = 1/c^2 dt^2 - \nabla^2$ $A^\mu = (\phi, \vec{A})$ is the electric and magnetic potential four-vector and $j^\mu = (\rho, \vec{j})$ is the charge/current density four-vector.

Plane wave solutions can be written as $A^\mu = \epsilon^\mu(s) e^{-ip\cdot x}$. $\epsilon^\mu(s)$ is the polarisation vector, which depends on the spin, s, of the photon.

6.4 Polarisation vectors

For spin-one bosons, there are three spin projections corresponding to three possible helicity states $s = +1, 0, -1$. $s = 0$ is known as *longitudinal* polarisation, and the $s = \pm 1$ are *transverse* polarisations (actually left and right-handed circular polarisations). For massless particles, the $s = 0$ state does not exist.

The equivalent of a fermion spinor is a **polarisation vector**, ϵ^μ, and the boson wavefunction is written:

$$\psi = \epsilon^\mu(p; s) e^{-ip\cdot x} \tag{6.8}$$

The polarisation vector is Lorentz gauge invariant:

$$p_\mu \epsilon^\mu = 0 \tag{6.9}$$

Virtual photons have $q^2 \neq 0$, and thus can have both longitudinal and transverse polarisations. This is also true for the massive W and Z bosons.

6.5 Feynman Rules for QED

Now we know how to describe photons and fermion currents, we can write down the Feynman rules for QED. These are shown in figure 6.1.

We use the Feynman Rules to write down the unique matrix element, $\mathcal{M}$, for the process. The matrix element represents the transition probability per unit time for that process to happen.

Write down the term for each piece of the Feynman diagram and multiply them all together to obtain $\mathcal{M}$:

- Start with a fermion line, and follow the arrows backwards. First write down the spinor for the (anti)fermion. Trace the arrow back to the vertex; write down the vertex term. Finally write down the spinor for the last part of the fermion line.

(If there is more than one vertex on the line, you will have to write down all the vertex term, and use the term for the internal fermion lines.) Do this for all the fermion lines.

- Write down the propagator term for the internal photons.

6.5.1 Summing Diagrams

If there is more than one Feynman diagram for a process, then you have to sum all the matrix elements:

$$\mathcal{M}_{\text{total}} = \mathcal{M}_1 + \mathcal{M}_2 + \cdots \tag{6.10}$$

In Fermi's Golden Rule (equation 4.2),we use the matrix element squared $|\mathcal{M}|^2$. When we sum diagrams:

$$|\mathcal{M}_{\text{total}}|^2 = (\mathcal{M}_1 + \mathcal{M}_2 + \cdots)(\mathcal{M}_1^* + \mathcal{M}_2^* + \cdots) \tag{6.11}$$

where $\mathcal{M}_1^*$ is the complex conjugate.

If we want to calculate the unpolarised cross section we need to **average over initial state spins and sum over final state spins**.

Remember we are using perturbation theory: each Feynman diagram represents only part of the process. Ideally we should sum all possible diagrams. However for each vertex the diagram is suppressed by $\sim 1/\sqrt{\alpha}$, therefore we often only need to consider only the lowest order diagrams (the ones with the least number of possible vertices). The most precise calculations in QED use diagrams with up to ten vertices, $\mathcal{O}(\alpha^5)$.

6.6 High energy $e^-\mu^- \to e^-\mu^-$ scattering

The process $e^-\mu^- \to e^-\mu^-$ is similar to spinless electromagnetic scattering in section 4.7, apart from the addition of the fermion spinors. The Feynman diagram, showing the spinor, vertex and propagator terms, is shown in figure 6.2.

The matrix element is written in terms of the two fermion currents:

$$\mathcal{M} = e^2 \frac{g_{\mu\nu}}{q^2} (\bar{u}_3 \gamma^\mu u_1)(\bar{u}_4 \gamma^\nu u_2) \tag{6.12}$$

Where u_i are the spinors for the incoming and outgoing particles, $i = 1, 2, 3, 4$.

The $g_{\mu\nu}$ allows us to change (contract) the γ^ν in the second bracket into γ^μ:

$$\mathcal{M} = \frac{e^2}{q^2} (\bar{u}_3 \gamma^\mu u_1)(\bar{u}_4 \gamma^\mu u_2) \tag{6.13}$$

For the unpolarised matrix element squared we must *average* over the initial spin states,

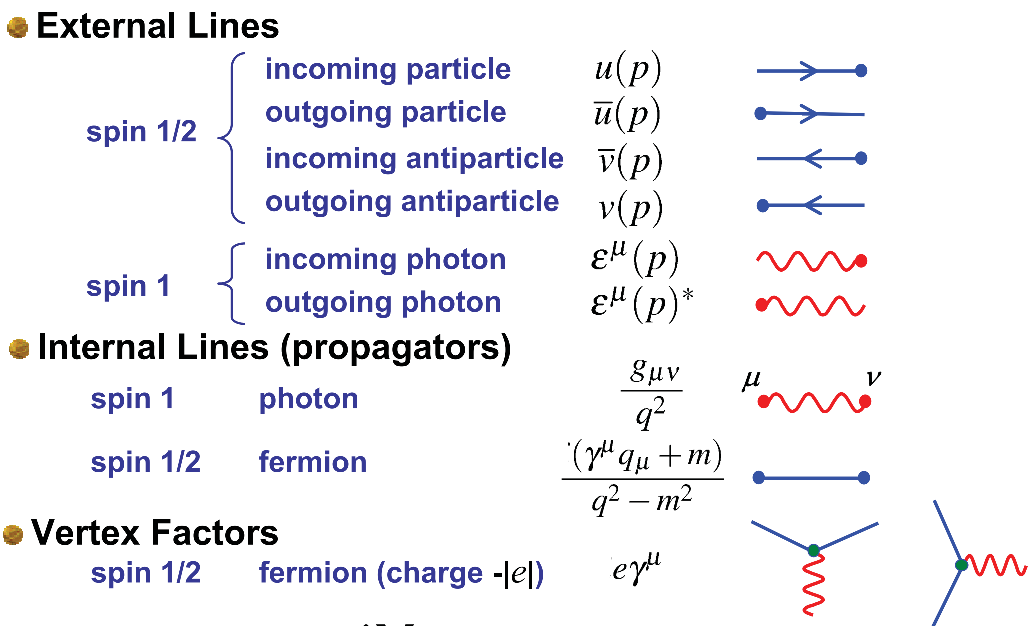


Figure 6.1: Feynman rules for QED.

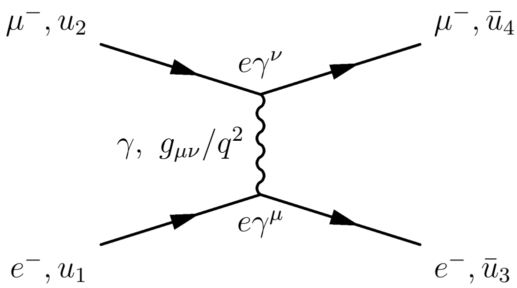


Figure 6.2: Feynman diagram for $e^- \mu^- \to e^- \mu^-$. The spinor, vertex and propagator terms used to calculate $\mathcal{M}$ are given.

and *sum* over the final spin states. We separate out the electron and muon parts:

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{e^4}{q^4} \frac{1}{(2S_1+1)(2S_2+1)} \sum_{S_3,S_4} (\bar{u}_3 \gamma^\mu u_1) (\bar{u}_3 \gamma^\nu u_1)^* (\bar{u}_4 \gamma^\mu u_2)(\bar{u}_4 \gamma^\nu u_2)^* \\
&= \frac{e^4}{q^4} \left(\frac{1}{(2S_1+1)} \sum_{S_3} (\bar{u}_3 \gamma^\mu u_1) (\bar{u}_3 \gamma^\nu u_1)^* \right) \left(\frac{1}{(2S_2+2)} \sum_{S_4} (\bar{u}_4 \gamma^\mu u_2)(\bar{u}_4 \gamma^\nu u_2)^* \right) \\
&= \frac{e^4}{q^4} L_e \, L_m \qquad (6.14)
\end{aligned}$$

where:

$$L_e = \frac{1}{(2S_1+1)} \sum_{S_3} (\bar{u}_3 \gamma^\mu u_1) (\bar{u}_3 \gamma^\nu u_1)^* \qquad (6.15)$$

where the initial electron spin is $S_1 = 1/2$, and the final spin states are S_3. For L_m replace $1 \to 2, 3 \to 4$. The calculation of the sum over the products of the spinors and gamma matrices looks horrible, but there is a trick using *trace theorems* which is discussed in detail on P.254/255 of Griffiths. The most relevant ones are:

$$Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu} \qquad Tr(\gamma_\mu \not{p}_1 \not{p}_3 \gamma^\mu) = 4(p_1 \cdot p_3) \tag{6.16}$$

with the conventional abbreviation $\not{p} = \gamma \cdot p = \gamma^\mu p_\mu = \sum_{\mu=0}^{3} \gamma^\mu p_\mu$. The electron sum gives:

$$L_e^{\mu\nu} = 2[p_3^\mu p_1^\nu + p_3^\nu p_1^\mu - (p_3 \cdot p_1 - m_e^2) g^{\mu\nu}] \tag{6.17}$$

and similarly for the muon part. Notice that there are no spinors left: the calculation is now just four vectors and matrices multiplied together. For high energy scattering the lepton masses (m_e, m_μ) can be neglected, and the matrix element squared is:

$$|\mathcal{M}|^2 = 8\frac{e^4}{q^4}[(p_3 \cdot p_4)(p_1 \cdot p_2) + (p_3 \cdot p_2)(p_1 \cdot p_4)] = 2e^4 \frac{s^2 + u^2}{t^2} \tag{6.18}$$

This differs from the spinless result which has $(s-u)^2$ rather than $s^2 + u^2$.

For polarised cross sections there are $2^4 = 16$ possible spin configurations, but only 4 of these are allowed at high energy, because helicity change at the vertices is suppressed. The four polarised matrix elements are:

$$\mathcal{M}(\uparrow\uparrow\uparrow\uparrow) = \mathcal{M}(\downarrow\downarrow\downarrow\downarrow) = e^2 \frac{u}{t} \qquad \mathcal{M}(\uparrow\downarrow\uparrow\downarrow) = \mathcal{M}(\downarrow\uparrow\downarrow\uparrow) = e^2 \frac{s}{t} \tag{6.19}$$

which gives the unpolarised result when the squares are summed.

6.6.1 $e^-\mu^- \to e^-\mu^-$ Cross Section

To calculate the cross section, we can use equation (4.16):

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\mathcal{S}|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\vec{p}_f^{\,*}|}{|\vec{p}_i^{\,*}|}$$

Making the following substitutions:

- centre of mass energy, $(E_1 + E_2)^2 = s$
- $|\vec{p}_f^{\,*}| = |\vec{p}_i^{\,*}|$ for elastic scattering
- $\mathcal{S} = 1$ as no identical particles in the final state
- $\alpha = e^2/(4\pi)$
- $|\mathcal{M}|^2$ from equation (6.18)

gives,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2\pi s} \left(\frac{s^2 + u^2}{t^2}\right) \tag{6.20}$$

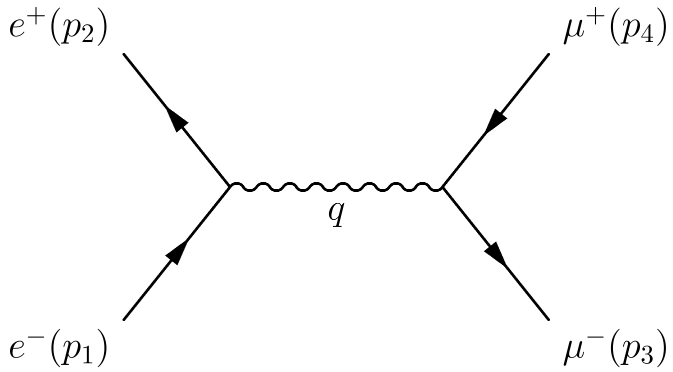


Figure 6.3: Lowest order Feynman diagram for $e^+e^- \to \mu^+\mu^-$

6.7 $e^+e^- \to \mu^+\mu^-$ Annihilation

The process $e^+e^- \to \mu^+\mu^-$ proceeds through the annihilation of the electron and positron into a virtual photon. The lowest order Feynman diagram is shown in figure 6.3.

The matrix element is:

$$\mathcal{M} = \frac{e^2}{q^2}\left(\bar{v}_2\gamma^\mu u_1\right)\left(\bar{u}_3\gamma_\mu v_4\right) \tag{6.21}$$

As above there are 16 possible helicity configurations but only 4 remain in the high energy limit, corresponding to annihilation/creation of fermion/antifermion pairs with opposite helicity states:

$$\mathcal{M}(\uparrow\downarrow\uparrow\downarrow) = \mathcal{M}(\downarrow\uparrow\downarrow\uparrow) = e^2\frac{u}{s} = \frac{e^2}{2}(1+\cos\theta^*) \tag{6.22}$$

$$\mathcal{M}(\uparrow\downarrow\downarrow\uparrow) = \mathcal{M}(\downarrow\uparrow\uparrow\downarrow) = e^2\frac{t}{s} = \frac{e^2}{2}(1-\cos\theta^*) \tag{6.23}$$

where θ is the scattering angle in the centre-of-mass system. It might be useful to think of this using the spin of the photon, $S = 1$. In the relativistic limit, helicity (or spin) is conserved, therefore the spin of the two incoming particles must add up to one: we must have one electron with $h = +1$ and one with $h = -1$. Likewise the two muons will be produced such that one has $h = +1$ and one has $h = -1$.

Note that $e^-\mu^- \to e^-\mu^-$ and $e^+e^- \to \mu^+\mu^-$ are related by a *crossing symmetry*. The matrix elements are related by the exchange of $s \leftrightarrow t$.

The total unpolarised matrix element squared for $e^+e^- \to \mu^+\mu^-$ is:

$$|\mathcal{M}|^2 = 2e^4\frac{t^2+u^2}{s^2} = e^4\left(1+\cos^2\theta^*\right) \tag{6.24}$$

Using equation (4.16), or by $t \leftrightarrow s$ in equation (6.20), the differential cross section in the centre-of-mass system is:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2\theta\right) \tag{6.25}$$

To get the total cross section we integrate over the solid angle, $d\Omega = d\cos\theta d\phi$:

$$\begin{aligned} \sigma &= \int \frac{d\sigma}{d\Omega} d\Omega = \frac{\alpha^2}{4\pi s} \int (1+\cos^2\theta) d\cos\theta d\phi \\ &= \frac{\alpha^2}{4\pi s} [\phi]_{-\pi}^{\pi} \left[\cos\theta + \frac{1}{3}\cos^3\theta\right]_{\cos\theta=-1}^{\cos\theta=+1} = \frac{4\alpha^2}{3s} \end{aligned} \tag{6.26}$$

The total cross section is proportional to $1/s$. Figure 6.4 shows measurements of this cross section. The agreement is pretty good, given that this we only used 1st order in pertubation theory!

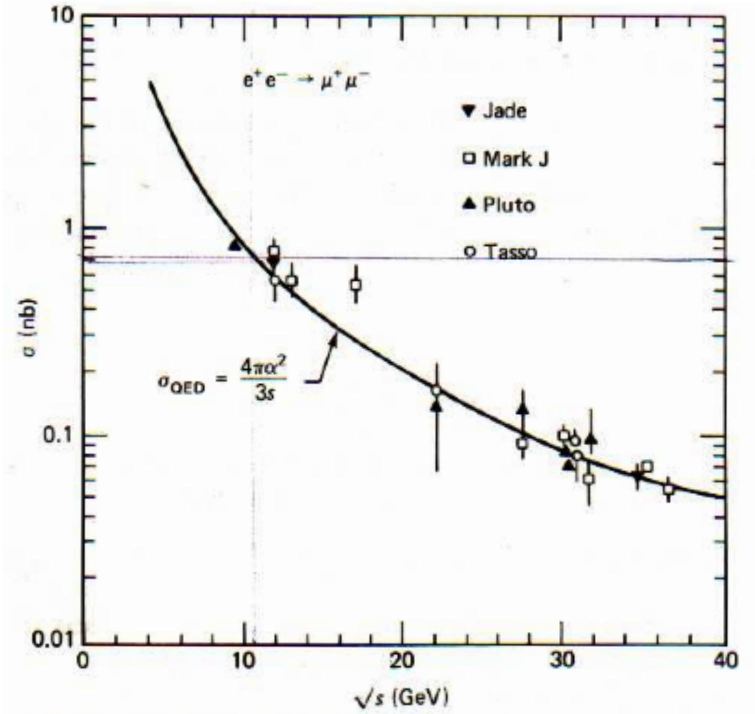


Fig. 6.6 **The total cross section for $e^-e^+ \to \mu^-\mu^+$ measured at PETRA versus the center-of-mass energy.**

Figure 6.4: Measured total cross section for $e^+e^- \to \mu^+\mu^-$ as a function of centre of mass energy, s, compared to our prediction.

6.8 Matrix elements for QED processes

A Table of QED matrix elements in the **high energy** limit from P.129 of Halzen & Martin are given in figure 6.5.

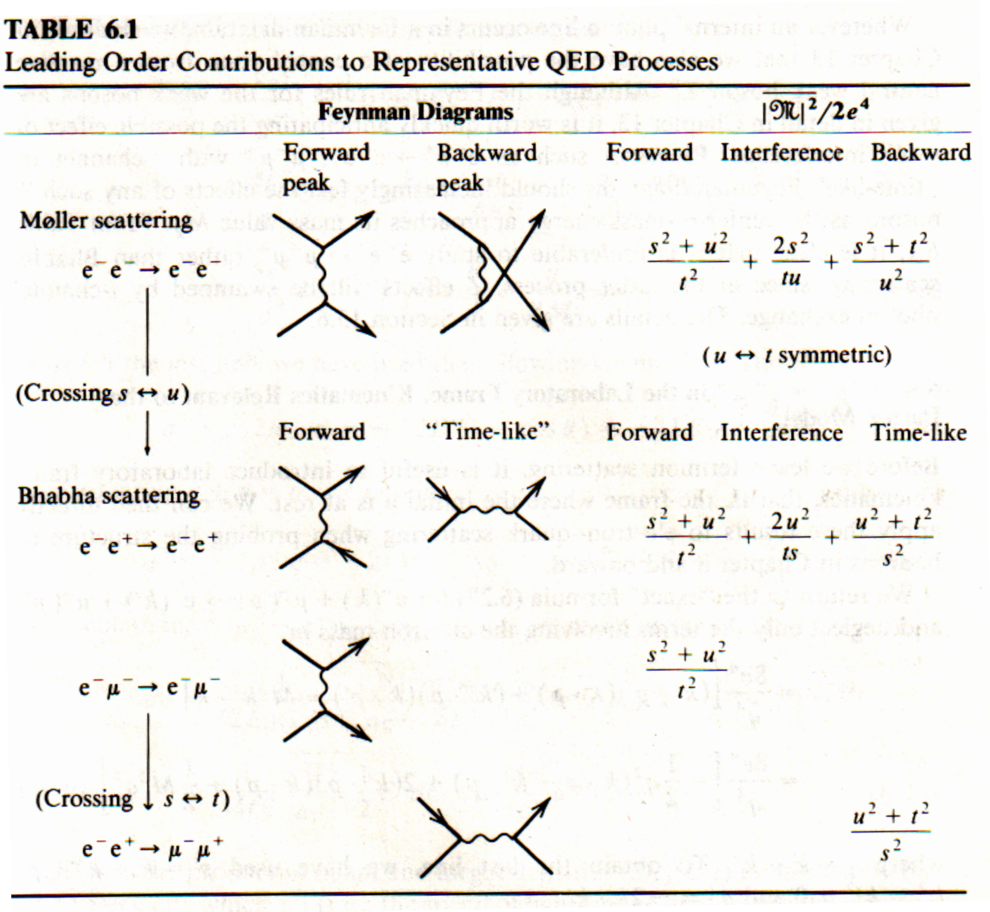
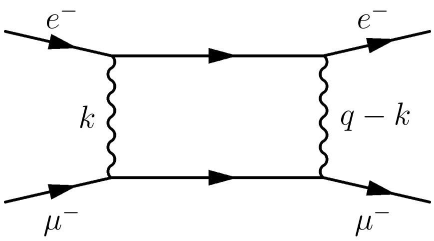


TABLE 6.1
Leading Order Contributions to Representative QED Processes

	Feynman Diagrams		$\overline{\lvert\mathcal{M}\rvert^2}/2e^4$		
	Forward peak	Backward peak	Forward	Interference	Backward
Møller scattering $e^-e^- \to e^-e^-$			$\dfrac{s^2+u^2}{t^2}$ +	$\dfrac{2s^2}{tu}$ +	$\dfrac{s^2+t^2}{u^2}$
			($u \leftrightarrow t$ symmetric)		
(Crossing $s \leftrightarrow u$)					
	Forward	"Time-like"	Forward	Interference	Time-like
Bhabha scattering $e^-e^+ \to e^-e^+$			$\dfrac{s^2+u^2}{t^2}$ +	$\dfrac{2u^2}{ts}$ +	$\dfrac{u^2+t^2}{s^2}$
$e^-\mu^- \to e^-\mu^-$			$\dfrac{s^2+u^2}{t^2}$		
(Crossing $s \leftrightarrow t$)					
$e^-e^+ \to \mu^-\mu^+$					$\dfrac{u^2+t^2}{s^2}$

Figure 6.5: QED matrix elements for lowest order processes in terms of the Mandelstam variables.



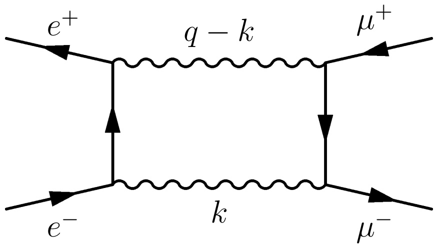


Figure 7.1: Higher order box diagrams with two photon exchange. Left: $e^-\mu^- \to e^-\mu^-$ scattering. Right: $e^+e^- \to \mu^+\mu^-$.

7 Renormalisation and the Weak Force

7.1 Higher Order Diagrams

As shown in figure 7.1 second photon can be exchanged we have considered above $e^-\mu^- \to e^-\mu^-$ scattering or in $e^+e^- \to \mu^+\mu^-$. These are examples of a higher order diagram known as a "box" diagram. There are two additional vertices, so the amplitude is reduced by a factor $\alpha = 1/137$ compared to the lowest order diagram with one photon exchange.

The overall four-momentum transfer is still q, but it has to be divided between the two photons with the first photon taking k. In fact k does not have to be less than q. It can have any value, including much larger than q, or opposite in sign. The fermion propagators between the two photons are also modified by k. It is simplest to think of the four momentum k as something that flows round the box in a clockwise or anticlockwise direction. Since it is only present for the virtual particles it is unobservable.

When calculating the amplitude for these diagrams it is necessary to perform an integral over the unknown four momentum k over a range which goes from zero to infinity. This integral diverges logarithmically. It took two decades of the twentieth century for Feynman and others to work out how to deal with these divergences, and obtain physically meaningful results for higher order diagrams.

> "All the infinities are miraculously swept up into formal expressions for quantities like physical mass and charge of the particles. These formally infinite expressions are then replaced by their finite physical values." (Aitchison & Hey P.51)

This process is known as **renormalization**.

These other higher order diagrams are also smaller than the lowest order diagram by a factor $\alpha = 1/137$. They all contain a loop with an arbitrary four momentum k that modifies all the propagators around the loop. In each case the integral over k leads to logarithmic infinities in the amplitudes which have to be removed by renormalization.

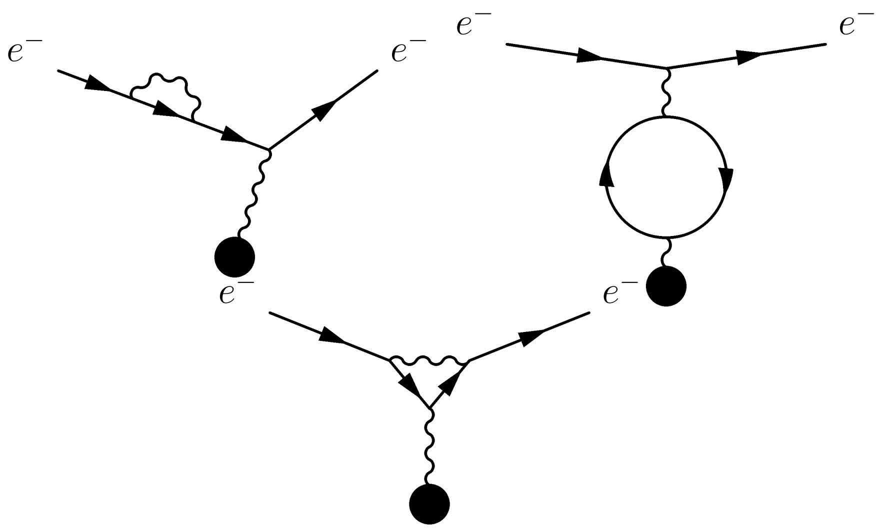


Figure 7.2: Some other higher order diagrams. Top left: a "dressed fermion" emits and reabsorbs a virtual photon. Top right: a "bubble" diagram where a virtual photon creates and annihilates a fermion-antifermion pair. Bottom: a "vertex correction" where a virtual photon connects two fermion lines across a vertex. The black blob represents an electromagnetic vertex which is not described in full.

7.2 Renormalization

In QED the divergent terms from higher order contributions to the amplitude are absorbed into a redefinition of the **charge** $e = \sqrt{4\pi\alpha}$ responsible for the vertex coupling, and a redefinition of the **fermion mass** m which enters through fermion propagators:

$$e_R = e_0 + \delta e \qquad m_R = m_0 + \delta m \tag{7.1}$$

The quantities e_0 and m_0 are known as the "bare" charge and mass. These would be the physical values if there were no higher order diagrams. Since there are always higher order diagrams, the bare values are unmeasurable. The remormalised physical values e_R and m_R are the ones we measure: they include δe and δm corrections from higher order diagrams, even though these are infinite! Rather bizarrely there must be cancellations between the infinities in δe, δm and the bare quantities e_0, m_0.

The renormalization process can be explicitly shown by imposing a "cutoff" mass M on the integral over the loop four momentum k. The integral then gives a finite part independent of M, and a part dependent on M which goes to infinity when the cutoff is removed (see Griffiths P. 219, P.264 and Halzen & Martin P. 157). The renormalized charge absorbs the infinite M dependent part (here we have just included electron-positron loops):

$$e_R = e_0 \sqrt{1 - \frac{e_0^2}{12\pi^2} ln\left(\frac{M^2}{q^2}\right)} \tag{7.2}$$

Taking a finite scale for M leads to a variation of the physical charge e as a function of the four momentum transfer q^2. This "running coupling constant" is discussed in the next section.

A way to understand renormalization is to say that the higher order contributions are absorbed into re-definitions of the Feynman diagrams:

- A dressed fermion contribution can be absorbed into a re-definition of the spinor, modifying the fermion mass.
- A bubble diagram can be absorbed into a re-definition of the photon propagator. This introduces lepton-antilepton and quark-antiquark components into the photon!
- A vertex correction can be absorbed into a re-definition of the coupling charge at the vertex. This can be thought of as being a result of charge screening due to fermion/antifermion pairs.

7.3 Running Coupling Constants

The finite parts of the higher order corrections lead to a q^2 dependence of the vertex couplings. Starting from the renormalized physical charge at $q^2 = 0$ the "running" coupling constant is:

$$\alpha(q^2) = \alpha(0)\left(1 + \alpha(0)\frac{z_f}{3\pi}ln(\frac{q^2}{M^2})\right) \tag{7.3}$$

where $z_f = \sum_f Q_f^2$ is the sum over the charges of all the active fermion/antifermion pairs (in units of e). z_f has a dependence on q^2 as more fermions become active. At low q^2 we can use $z_f = 1$ (only e^+e^- pairs), but at 100 GeV we need $z_f = 38/9 \approx 4$ (only $t\bar{t}$ pairs not active).

Since M is an arbitrary large cut-off mass, the above form is not particularly useful. What is usually done is to select a **renormalization scale**, $q^2 = \mu^2$, relative to which α at any other value of q^2 can be defined:

$$\alpha(q^2) = \alpha(\mu^2)\left(1 - \alpha(\mu^2)\frac{z_f}{3\pi}ln(\frac{q^2}{\mu^2})\right)^{-1} \tag{7.4}$$

We can choose any value of μ where make a initial measurement of α, but once we do the evolution of the values of α are determined by equation (7.4). It is usual to take either a low energy reference from atomic physics ($\mu = 1$ MeV) or a high energy reference from LEP ($\mu = m_Z = 91\ GeV$)data:

$$\alpha(\mu = 0) = \frac{1}{137} \qquad \alpha(\mu^2 = m_Z^2) = \frac{1}{128} \tag{7.5}$$

Note that the running of the electromagnetic coupling α is quite small. We will see later that this is not the case for the strong coupling α_s.

7.4 Measurements of $g - 2$

Non-Examinable section - just included for your interest

The gyromagnetic ratio, g, for an electron, defines its magnetic moment, i.e. the coupling of its spin to a magnetic field:

$$\vec{\mu} = g\mu_B\vec{S} \qquad \mu_B = \frac{e\hbar}{2m_e c} \tag{7.6}$$

where μ_B is the Bohr magneton. One of the successes of the Dirac equation is that it predicts that $g = 2$ for "bare" fermions.

The lowest order Feynman diagram describing the magnetic moment couples the fermion to the electromagnetic field via a single virtual photon. The inclusion of higher order diagrams lead to an *anomalous* value for g slightly different from 2. QED has been used to calculate diagrams up to $\mathcal{O}(\alpha^5)$, of which there are a very large number! The theoretical predictions for the electron and muon are:

$$\left[\frac{g-2}{2}\right]_e = 0.001159652183(8) \qquad \left[\frac{g-2}{2}\right]_\mu = 0.0011659183(5) \tag{7.7}$$

where the numbers in brackets are the errors on the last digits. *Note that the main theoretical uncertainties are no longer associated with QED but with the introduction of strong interaction effects when a bubble diagram involves a quark-antiquark pair.*

The experimental measurements of $g-2$ use spin precession in a magnetic field. The electron is stable and can be held in a Penning trap to make the measurements. In the case of the unstable muon, there has been a heroic series of experiments over the last 50 years using storage rings. The current results are:

$$\left[\frac{g-2}{2}\right]_e = 0.0011596521807(3) \qquad \left[\frac{g-2}{2}\right]_\mu = 0.0011659209(6) \tag{7.8}$$

These tests of QED are among the most accurate tests of a theory in the whole of physics. There is quite a lot of current interest in the $(26 \pm 8) \times 10^{-10}$ difference between experiment and theory for the muon which has about 3σ significance. This may be evidence for physics beyond the Standard Model giving rise to additional Feynman diagrams. *End of non-examinable section*

7.5 Charged Currents

The $W^\pm$ bosons with mass $M_W = 80.4$ GeV mediate weak charged current interactions:

- The heavy virtual W boson propagator is $g^{\mu\nu}/(M_W^2 - q^2)$.
- The current operator is purely left-handed $\gamma^\mu \frac{1}{2}(1-\gamma^5)$.
  This is known as the **V–A theory**.
- The dimensionless coupling constant at each vertex, g_W, is often written in terms of the dimensioned **Fermi constant**, $G_F = 1.16 \times 10^{-5}\text{GeV}^{-2}$:

  $$\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8M_W^2} \tag{7.9}$$

  This comes from the low energy limit on the propagator $q^2 \ll M_W^2$.

Figure 7.3: Quark level β decay Feynman diagrams.

- The $W^\pm$ couplings to leptons conserve lepton family number: $W^+ \to e^+\nu_e$, $\mu^+\nu_\mu$, $\tau^+\nu_\tau$ and $W^- \to e^-\bar{\nu}_e$, $\mu^-\bar{\nu}_\mu$, $\tau^-\bar{\nu}_\tau$.

- The $W^\pm$ couplings to quarks *change quark flavour*. All possible $\mathrm{u}_i\bar{\mathrm{d}}_j$ or $\bar{\mathrm{u}}_i d_j$ pairs are allowed, where $\mathrm{u}_i = (u, c, t)$, $\mathrm{d}_j = (d, s, b)$.

- The coupling constants for quarks are gV_{ij} where V_{ij} is the 3×3 Cabibbo-Kobayashi-Maskawa matrix.

$$V_{CKM} = \begin{pmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{pmatrix} \tag{7.10}$$

7.6 Beta decay

Inside a nucleus weak charged current interactions transform a proton into a neutron (β^+ decays), or a neutron into a proton (β^- decays):

$$^{14}\mathrm{O} \to ^{14}\mathrm{N}^* + e^+ + \nu_e \qquad ^{60}\mathrm{Co} \to ^{60}\mathrm{Ni}^* + e^- + \bar{\nu}_e \tag{7.11}$$

A free neutron decays to a proton, electron and antineutrino, with a lifetime of 886 s, but a free proton does not decay to a neutron. This is because the proton mass, 938.3 MeV, is lower than the neutron mass, 939.6 MeV. At the quark level β decay is described by:

$$u \to d + e^+ + \nu_e \qquad d \to u + e^- + \bar{\nu}_e \tag{7.12}$$

If the electron (positron) is transferred from the final to the initial state we have **electron capture**:

$$e^- + u \to d + \nu_e \qquad e^+ + d \to u + \bar{\nu}_e \tag{7.13}$$

If the neutrino (antineutrino) is transferred from the final to the initial state we have **inverse β decay**:

$$\bar{\nu}_e + u \to d + e^+ \qquad \nu_e + d \to u + e^- \tag{7.14}$$

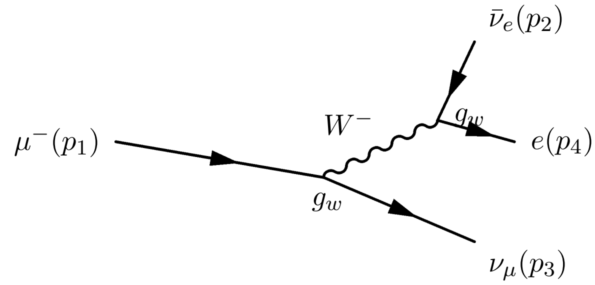
7.6.1 Matrix element for β decay

The matrix element for β decay can be written in terms of quark and lepton currents containing the fermion spinors:

$$\mathcal{M} = \left(\frac{g}{\sqrt{2}}\bar{u}_d\gamma^\mu\frac{1}{2}(1-\gamma^5)u_u\right)\frac{1}{M_W^2-q^2}\left(\frac{g}{\sqrt{2}}\bar{u}_{\nu_e}\gamma^\mu\frac{1}{2}(1-\gamma^5)u_e\right) \tag{7.15}$$

Hadronic interactions (form factors) play a role in decay rate and lifetimes, and the above formula need to be modifed for these effects.

7.7 Muon decay



The muon is a charged lepton that decays to an electron, a muon neutrino and an electron antineutrino:

$$\mu^- \to e^-\nu_\mu\bar{\nu}_e \qquad \mu^+ \to e^+\nu_e\bar{\nu}_\mu \tag{7.16}$$

The matrix element can be written in terms of a muon-type current between the muon and the muon neutrino, and an electron-type current between the electron and the electron antineutrino:

$$|\mathcal{M}|^2 = \left(\frac{g_W^2}{8m_W^2}\right)^2 [\bar{u}(\nu_\mu)\gamma^\mu(1-\gamma^5)u(\mu)]^2[\bar{u}(e)\gamma^\mu(1-\gamma^5)v(\bar{\nu}_e)]^2 \tag{7.17}$$

Neglecting the masses of the electron and neutrinos the transition rate can be calculated (see Griffiths P.311-313):

$$\frac{d\Gamma}{dE_e} = \frac{G_F^2}{4\pi^3}m_\mu^2E_e^2\left(1-\frac{4E_e}{3m_\mu}\right) \tag{7.18}$$

This is known as the Michel spectrum. Recent measurement of this are shown in figure 7.4. Near the endpoint $E_0 = m_\mu/2$ the rate drops abruptly from a maximum to zero.

We can obtain the decay rate by integrating equation (7.18):

$$\Gamma = \int_0^{m_\mu/2}\frac{d\Gamma}{dE_e}dE_e = \frac{G_F^2m_\mu^2}{4\pi^3}\int_0^{m_\mu/2}E_e^2\left(1-\frac{4E_e}{3m_\mu^2}\right)dE_e \tag{7.19}$$

As $\mu^- \to e^-\overline{\nu_e}\nu_\mu$ is the only possible decay, the lifetime of the muon is:

$$\tau \equiv \frac{1}{\Gamma} = \frac{192\pi^3}{G_F^2m_\mu^5} = \frac{192\pi^3\hbar^7}{G_F^2m_\mu^5c^4} \tag{7.20}$$

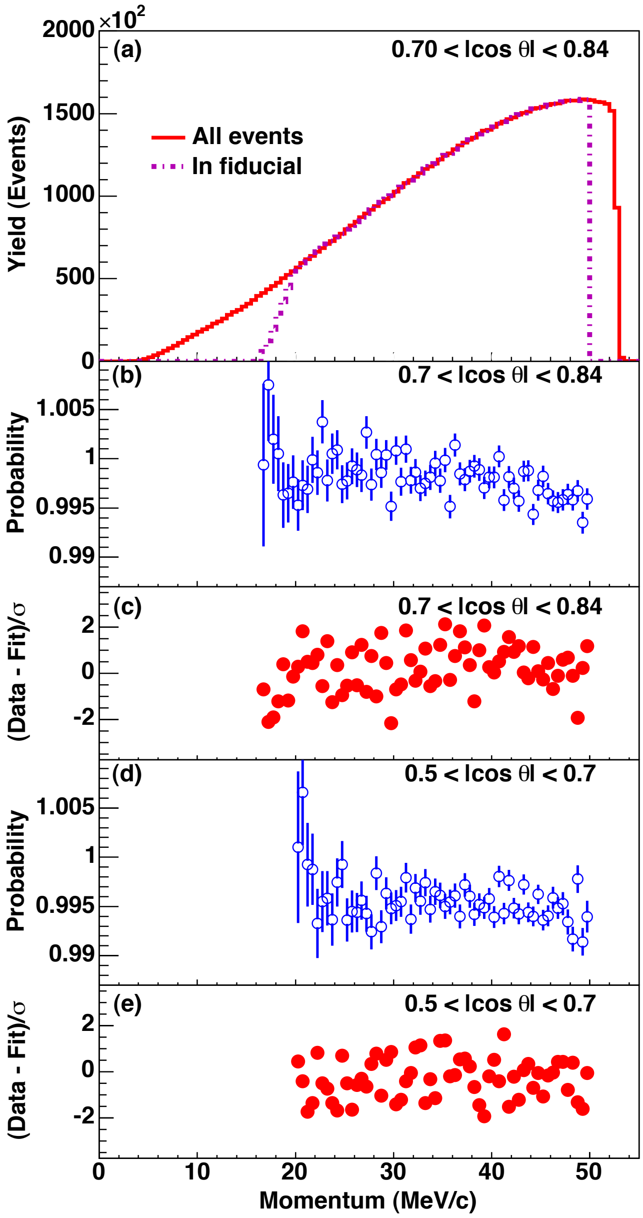


Figure 7.4: TWIST experiment at TRIMF in Canada measures $\mu^+ \to e^+\nu_e\overline{\nu_\mu}$ decay spectrum. Excellent agreement between data and prediction!

Measurements of muon lifetime and mass used to define a value for G_F (values from PDG 2010)

$$\tau = (2.19703 \pm 0.00002) \times 10^6 \text{ s} \qquad m = 105.658367 \pm 0.000004 MeV \tag{7.21}$$

Applying small corrections for finite electron mass and second order effects $G_F = 1.166364(5) \times 10^{-5}$ GeV^{-2}

We can compare the couplings in muon decay and superallowed β decay:

$$G_F(\mu) = 1.166 \times 10^{-5} \text{ GeV}^{-2} \qquad G_F(\beta)|V_{ud}| = (1.136 \pm 0.003) \times 10^{-5} \text{ GeV}^{-2}$$

These give the same value for G_F with $|V_{\rm ud}| \approx 0.97$.

7.8 Neutral Currents

The Z^0 boson with mass $M_Z = 91.2$ GeV mediates weak neutral current interactions:

- The heavy virtual Z boson propagator is $1/(M_Z^2 - q^2)$.
- The current operator is no longer purely left-handed except for neutrinos. It is written as $\gamma^\mu(c_V - c_A\gamma^5)$, where $g_L = (c_V + c_A)/2$ and $g_R = (c_V - c_A)/2$, and $c_V(c_A)$ are the vector (axial-vector) couplings.
- The Z^0 couplings to leptons conserve lepton flavour:
  $Z^0 \to e^+e^-$, $\mu^+\mu^-$, $\tau^+\tau^-$ and $Z^0 \to \nu_e\bar{\nu}_e$, $\nu_\mu\bar{\nu}_\mu$, $\nu_\tau\bar{\nu}_\tau$.
- The Z^0 couplings to quarks conserve quark flavour:
  $Z^0 \to u\bar{u}$, $c\bar{c}$, $d\bar{d}$, $s\bar{s}$, $b\bar{b}$, $(t\bar{t})$
- The coupling constant at each vertex is g, but modified by the values of c_V and c_A which depend on fermion type.

Lepton	$2c_V$	$2c_A$	Quark	$2c_V$	$2c_A$
ν_e, ν_μ, ν_τ	1	1	u, c, t	0.38	1
e, μ, τ	-0.06	-1	d, s, b	-0.68	-1

At low centre-of-mass energies the couplings of the Z^0 to quarks and charged leptons are a small addition to the electromagnetic couplings, which can only be observed in precision experiments. At higher energies, the interference between Z^0 and γ exchange is important. It is explained in terms of the full electroweak theory.

There is one place where the neutral current couplings to Z^0 can be observed unambiguously. This is in neutrino scattering, since neutrinos have no charge and do not experience the electromagnetic interaction. The first observation of these couplings was at CERN in 1973.